

A possible symplectic framework for Radon-type transforms

Michel Cahen¹, Thibaut Grouy and Simone Gutt¹

Département de Mathématiques, Université Libre de Bruxelles

boulevard du triomphe, Campus Plaine CP 218

B-1050 Brussels, Belgium

¹Académie Royale de Belgique

mcahen@ulb.ac.be, thibautgrouy@gmail.com, sgutt@ulb.ac.be

Abstract

Our project is to define Radon-type transforms in symplectic geometry. The chosen framework consists of symplectic symmetric spaces whose canonical connection is of Ricci-type. They can be considered as symplectic analogues of the spaces of constant holomorphic curvature in Kählerian Geometry. They are characterized amongst a class of symplectic manifolds by the existence of many totally geodesic symplectic submanifolds. We present a particular class of Radon type tranforms, associating to a smooth compactly supported function on a homogeneous manifold M , a function on a homogeneous space N of totally geodesic submanifolds of M , and vice versa. We describe some spaces M and N in such Radon-type duality with M a model of symplectic symmetric space with Ricci-type canonical connection and N an orbit of totally geodesic symplectic submanifolds.

Introduction

The subject of Radon transforms started with results by Funk, in 1913, who observed that a symmetric function on the sphere S^2 can be described from its great circle integrals, and by Radon, in 1917, who showed that a smooth function f on the Euclidean space \mathbb{R}^3 can be determined by its integrals over the planes in \mathbb{R}^3 : if $J(\omega, p)$ is the integral of f over the plane defined by $x \cdot \omega = p$ for ω a unit vector, then

$$f(x) = -\frac{1}{8\pi^2} L_x \left(\int_{S^2} J(\omega, \omega \cdot x) d\omega \right)$$

where L is the Laplacian. This underlines a correspondence between the space $M = \mathbb{R}^3$ and the space of 2-planes in \mathbb{R}^3 , $N = Pl(2, \mathbb{R}^3) = (S^2 \times \mathbb{R})/\mathbb{Z}_2$:

- the Radon transform associates to a smooth compactly supported function f on \mathbb{R}^3 , the function $\text{Rad } f$ on $Pl(2, \mathbb{R}^3)$ defined by

$$\text{Rad } f(S) = \int_{x \in S} f(x) dm(x) \quad \text{for any 2-plane } S;$$

where dm is the Lebesgue measure on S .

- the dual Radon transform associates to a smooth compactly supported function φ on $Pl(2, \mathbb{R}^3)$,

the function $\text{Rad}^*\varphi$ on \mathbb{R}^3 defined by :

$$\text{Rad}^*\varphi(x) = \int_{S \ni x} \varphi(S) d\mu(S)$$

where $d\mu$ is the unique normalized measure on $Pl(2, \mathbb{R}^3)$ which is invariant under rotations. Observing that one can recover a compactly supported function on \mathbb{R}^3 from its integral over lines in \mathbb{R}^3 led to applications to X-ray technology and tomography and boosted the interest in this theory. The idea of the Radon transform was widely generalized, constructing correspondances between a class of objects on a space M and a class of objects on a space N .

We consider only Radon-type transforms between spaces of functions built by integrating functions on totally geodesic submanifolds in a homogeneous framework. This was introduced by Helgason, mostly in a (pseudo) Riemannian framework, in the sixties. Let us recall that a connected submanifold S of a manifold M endowed with a connection ∇ is said to be *totally geodesic* if each geodesic in M which is tangent to S at a point lies entirely in S . These submanifolds have been described for pseudo-Riemannian space forms (M, g) (endowed with the Levi Civita connection). The abstract Radon transform generalizes the original transform, associating to a smooth compactly supported function f on a homogeneous space M the function $\text{Rad} f$ on a homogeneous space N of totally geodesic submanifolds of fixed dimension p , whose value at a point $S \in N$ is given by the integral of f on the corresponding submanifold S

$$f \rightarrow \text{Rad} f(S) = \int_{x \in S} f(x) dm(x). \quad (1)$$

The dual abstract Radon transform is :

$$\varphi \rightarrow \text{Rad}^*\varphi(x) = \int_{S \ni \{x\}} \varphi(S) d\mu(S). \quad (2)$$

The first questions encountered in constructing such Radon-type transforms deal with finding invariant measures to integrate the functions, relating functions spaces, inverting the transforms, finding a corresponding map between invariant differential operators, studying the support of f when $\text{Rad} f$ has compact support... They have been studied in the framework of Riemannian spaces of constant curvature, which provide a rich supply of totally geodesic submanifolds. The results and the original references on the subject are presented in Helgason's nice books [8, 9].

To build such correspondences in a symplectic framework, we choose symplectic symmetric spaces with Ricci-type canonical connection. We recall their definition in section 1, along with their properties and the construction of models. In section 2, we describe the space of totally geodesic symplectic submanifolds of our models. The corresponding list of homogeneous spaces in symplectic Radon duality was announced in [7] when the Ricci endomorphism of the tangent bundle squares to a non zero constant multiple of the identity. We extend this list to the case where it squares to zero.

In section 3, we characterize the symplectic locally symmetric spaces with Ricci-type canonical connection of dimension ≥ 8 as the symplectic manifolds with a symplectic connection (M, ω, ∇) having the 2 following properties :

1. they admit a parallel field A of endomorphisms of the tangent bundle such that $A_x \in \mathfrak{sp}(T_x M, \omega_x)$ and $A^2 = \lambda \text{Id}$;
2. given any point x and any A_x -stable symplectic subspace V_x of $T_x M$, there is a unique maximal totally geodesic symplectic submanifold passing through x and tangent to V_x .

1 Ricci-type symplectic symmetric spaces

1.1 Symplectic connections

A linear connection ∇ on a symplectic manifold (M, ω) is called *symplectic* if the symplectic form ω is parallel and if its torsion T^∇ vanishes.

Let us recall that a symplectic connection exists on any symplectic manifold. Indeed, there is a canonical projection of any torsion free connection ∇^0 on a symplectic connection ∇ given as follows : define N by $\nabla_X^0 \omega(Y, Z) =: \omega(N(X, Y), Z)$ and set

$$\nabla_X Y := \nabla_X^0 Y + \frac{1}{3}N(X, Y) + \frac{1}{3}N(Y, X).$$

On the other hand a symplectic connection is far from being unique: given ∇ symplectic, the connection $\nabla'_X Y := \nabla_X Y + S(X, Y)$ is symplectic if and only if $\omega(S(X, Y), Z)$ is totally symmetric.

When the manifold is symmetric there is a natural unique connection: it is the canonical connection for which each symmetry is an affine transformation. In the symplectic context we have precisely:

A *symplectic symmetric space* is a symplectic manifold (M, ω) endowed with “symmetries” i.e. with a smooth map

$$S : M \times M \rightarrow M : (x, y) \mapsto s_x y$$

so that, for each $x \in M$, the symmetry $s_x : M \rightarrow M$ is an involutive symplectomorphism (i.e. $s_x^* \omega = \omega$ and $s_x^2 = \text{Id}$), with x an isolated fixed point (i.e. $s_x x = x$ and 1 is not an eigenvalue of $(s_x)_{*x}$), and such that $s_x s_y s_x = s_{s_x y}$, for any $x, y \in M$.

On a symplectic symmetric space there is a unique connection for which each s_x is an affine transformation; it is given by

$$(\nabla_X Y)_x = \frac{1}{2}[X - s_{x*} X, Y]_x. \quad (3)$$

This connection is automatically symplectic. Symmetric symplectic spaces were introduced in [2, 3].

The curvature tensor of a symplectic connection at a point x has the following symmetry properties:

$$\omega_x(R_x^\nabla(X, Y)Z, T) = -\omega_x(R_x^\nabla(Y, X)Z, T) = \omega_x(R_x^\nabla(X, Y)T, Z)$$

and

$$\bigoplus_{XYZ} R_x^\nabla(X, Y)Z = 0 \quad , \quad \bigoplus_{XYZ} \text{ denoting the sum over cyclic permutations of } X, Y \text{ and } Z.$$

When $\dim M = 2n \geq 4$, Izu Vaisman [12] showed that the space of tensors having those symmetries splits under the action of the symplectic group into two irreducible components so that one has a decomposition of the curvature into

$$R^\nabla = W^\nabla + E^\nabla \quad (4)$$

where W^∇ has no trace and

$$\begin{aligned} E^\nabla(X, Y)Z &= \frac{1}{2n+2} (2\omega(X, Y)\rho^\nabla Z + \omega(X, Z)\rho^\nabla Y - \omega(Y, Z)\rho^\nabla X \\ &\quad + \omega(X, \rho^\nabla Z)Y - \omega(Y, \rho^\nabla Z)X) \end{aligned} \quad (5)$$

with the Ricci tensor r^∇ (defined by $r^\nabla(X, Y) := \text{Tr}[Z \rightarrow R^\nabla(X, Z)Y]$), which is automatically symmetric, converted into the Ricci endomorphism ρ^∇ by

$$\omega(X, \rho^\nabla Y) = r^\nabla(X, Y). \quad (6)$$

A symplectic connection for which $W^\nabla = 0$ is said to be of *Ricci-type*.

1.2 Space forms

Let us recall that a *Pseudo Riemannian space form* is a connected pseudo Riemannian manifold (N, \dot{g}) of dimension $n \geq 4$, which is geodesically complete (for the Levi Civita connection), and which has constant sectional curvature k .

Its curvature \dot{R} is then given by

$$\dot{g}_x(\dot{R}_x(X, Y)Z, T) = k(\dot{g}_x(X, Z)\dot{g}_x(Y, T) - \dot{g}_x(X, T)\dot{g}_x(Y, Z)); \quad (7)$$

the space (N, \dot{g}) is thus locally symmetric ($\nabla \dot{R} = 0$) and the curvature is a polynomial in the tensor algebra only involving the metric tensor \dot{g} .

Riemannian space forms in dimension n are quotients of the Euclidean space \mathbb{R}^n , the sphere S^n or the hyperbolic space H^n .

If a symplectic manifold (M, ω) is endowed with a symplectic connection whose curvature is a polynomial in the tensor algebra only involving the symplectic tensor ω , then the curvature is invariant under the symplectic group and, in view of the decomposition formula (4), it is identically zero. In the definition of symplectic space forms, we obviously want to go beyond the flat case, so we consider the Kähler case to generalize it.

A *Kähler Space form* is a connected Kähler manifold (N, \dot{g}, J) of dimension $n \geq 4$, which is geodesically complete (for the Levi Civita connection) and which has constant holomorphic sectional curvature k (i.e.: $\dot{g}_x(\dot{R}_x(X, JX)X, JX) = k(\dot{g}_x(X, X)^2)$ for all $x \in N$ and for all $X \in T_x N$).

Its curvature is then given by

$$\begin{aligned} \dot{g}_x(\dot{R}_x(X, Y)Z, T) = & \frac{k}{4} \left(\dot{g}_x(X, Z)\dot{g}_x(Y, T) - \dot{g}_x(X, T)\dot{g}_x(Y, Z) + \dot{g}_x(X, JZ)\dot{g}_x(Y, JT) \right. \\ & \left. - \dot{g}_x(X, JT)\dot{g}_x(Y, JZ) + 2\dot{g}_x(X, JY)\dot{g}_x(Z, JT) \right); \end{aligned} \quad (8)$$

the space is thus locally symmetric and the curvature is a polynomial in the tensor algebra (with catenation) involving only \dot{g} and J .

Kähler space forms in complex dimension n are all quotients of \mathbb{C}^n , the complex projective space \mathbb{CP}^n , or the complex hyperbolic space \mathbb{CH}^n .

Remark that the Kähler form $\omega_x(X, Y) := \dot{g}_x(X, JY)$ is symplectic and the Levi Civita connection is symplectic. In symplectic terms, formula (8) rewrites as $\dot{R}_x(X, Y)Z = \frac{k}{4}(-\omega_x(X, JZ)Y + \omega_x(Y, JZ)X - \omega_x(X, Z)JY + \omega_x(Y, Z)JX - 2\omega_x(X, Y)JZ)$. Hence the Levi Civita connection is of Ricci type with Ricci endomorphism equal to a multiple of the complex structure: $\dot{\rho} = -\frac{k(n+1)}{2}J$.

Definition 1.1. A **symplectic space form** is a connected symplectic manifold endowed with a symplectic connection ∇ which is complete, locally symmetric, and such that its curvature is of Ricci type.

The curvature in a symplectic space form is thus given by

$$\begin{aligned} R^\nabla(X, Y)Z = & \frac{1}{2n+2} \left(2\omega(X, Y)\rho^\nabla Z + \omega(X, Z)\rho^\nabla Y - \omega(Y, Z)\rho^\nabla X \right. \\ & \left. + \omega(X, \rho^\nabla Z)Y - \omega(Y, \rho^\nabla Z)X \right) \end{aligned} \quad (9)$$

with ρ^∇ the Ricci endomorphism. The condition to be locally symmetric is equivalent to $\nabla \rho^\nabla = 0$, thus ρ^∇ commutes with $R^\nabla(X, Y)$ and this yields $\omega(X, Z)(\rho^\nabla)^2 Y - \omega(Y, Z)(\rho^\nabla)^2 X = \omega(X, (\rho^\nabla)^2 Z)Y - \omega(Y, (\rho^\nabla)^2 Z)X$ which implies $(\rho^\nabla)^2 = k \text{Id}$ where k is a real constant.

Symmetric symplectic spaces with Ricci-type connections were studied with John Rawnsley in [4]; Nicolas Richard examined the analogue of the notion of constant holomorphic sectional curvature in a symplectic context in his thesis [11].

1.3 Models of Ricci-type symmetric symplectic spaces

We recall a construction by reduction of examples of Ricci-type symplectic connections which was given by Baguis and Cahen in [1]. Let $(\mathbb{R}^{2n+2} \setminus \{0\}, \Omega)$ be the standard symplectic vector space without the origin and let A be an element in the symplectic Lie algebra $sp(\mathbb{R}^{2n+2}, \Omega)$, i.e. ${}^{tr}A\Omega + \Omega A = 0$, where ${}^{tr}B$ denotes the transpose of the matrix B . We consider the reduction with respect to the action of the 1-parameter subgroup $\exp tA$; the action is Hamiltonian with corresponding Hamiltonian $h(x) = \frac{1}{2}\Omega(x, Ax)$ and we consider the embedded hypersurface given by a level set: $\Sigma_A = \{x \in \mathbb{R}^{2n+2} \mid \Omega(x, Ax) = 1\}$, assuming it to be non empty. The 1-parameter subgroup $\{\exp tA\}$ acts on Σ_A and we consider the quotient

$$M^{red} := \Sigma_A / \exp tA \quad (\text{it exists at least locally since } Ax \neq 0!)$$

with the canonical projection

$$\pi : \Sigma_A \rightarrow M^{red}.$$

For any $x \in \Sigma_A$, we define a $2n$ -dimensional “horizontal” subspace H_x of the tangent space $T_x \Sigma_A$ given by the Ω -orthogonal to the subspace spanned by x and Ax :

$$H_x := \text{Span}\{x, Ax\}^{\perp \Omega} \subset T_x \Sigma_A \simeq \text{Span}\{Ax\}^{\perp \Omega} \subset \mathbb{R}^{2n+2};$$

the differential of the projection π induces an isomorphism

$$\pi_{*x} : H_x \xrightarrow{\sim} T_{\pi(x)} M^{red}.$$

Given a tangent vector $X \in T_y M^{red}$, we denote by \overline{X} its horizontal lift:

$$\overline{X}_x \in H_x, \quad X_{y=\pi(x)} = \pi_{*x} \overline{X}_x.$$

The reduced 2-form ω^{red} on M^{red} is defined in the standard way as the unique 2-form ω^{red} on M^{red} such that :

$$\pi^* \omega^{red} = \Omega|_{\Sigma_A} \quad \text{i.e.} \quad \omega_{y=\pi(x)}^{red}(X, Y) := \Omega_x(\overline{X}_x, \overline{Y}_x); \quad (10)$$

and (M^{red}, ω^{red}) is a symplectic manifold. The flat connection $\overset{\circ}{\nabla}$ on \mathbb{R}^{2n+2} induces a connection ∇^{red} on M^{red} given by

$$(\nabla_X^{red} Y)_y := \pi_{*x} (\overset{\circ}{\nabla}_{\overline{X}} \overline{Y} - \Omega(A\overline{X}, \overline{Y})x + \Omega(\overline{X}, \overline{Y})Ax). \quad (11)$$

This reduced connection ∇^{red} on (M^{red}, ω^{red}) is symplectic and of Ricci-type. With Lorenz Schwachhöfer, we proved in [5] that any symplectic manifold endowed with a symplectic connection of Ricci-type is locally of this form. In the above construction the Ricci endomorphism at $y = \pi(x) \in M^{red}$ is proportional to the map induced by A on H_x :

$$-\frac{1}{2n+2} \rho_{\pi(x)}^{\nabla^{red}}(X) = \pi_{*x} (A\overline{X} - \Omega(A\overline{X}, Ax)x).$$

The space is locally symmetric if and only if $(\rho^\nabla)^2 = k \text{Id}$ and this happens if and only if $A^2 = \lambda \text{Id}$.

The models $(M_A, \omega^{red}, \nabla^{red})$ of symplectic space forms that we shall use are the connected components of the ones obtained by the above construction, using an element $A \in sp(\mathbb{R}^{2n+2}, \Omega)$ so that $A^2 = \lambda \text{Id}$; in these cases the quotient of the hypersurface Σ_A by the action of $\{\exp tA\}$ is globally defined and we have

$$\pi : \Sigma_A = \{x \in \mathbb{R}^{2n+2} \mid \Omega(x, Ax) = 1\} \rightarrow M_A := M_{cc}^{red} = (\Sigma_A / \{\exp tA\})_{cc}$$

where $_{cc}$ indicates that we take a connected component.

Any $B \in \tilde{G}_A := \{B \in Sp(\mathbb{R}^{2n+2}, \Omega) \mid BA = AB\}_0$, where $_0$ denotes the connected component of the identity, induces an automorphism of M_A in the obvious way:

$$B \cdot \pi(x) := \pi(Bx).$$

The space M_A is not only locally symmetric; it is a symplectic symmetric space, with the symmetry at $y = \pi(x)$ induced by

$$S_x(v) = -v - 2\Omega(v, x)Ax + 2\Omega(v, Ax)x.$$

Theorem 1.2. [5, 6] *Let $A \in sp(\mathbb{R}^{2n+2}, \Omega)$ be so that $A^2 = \lambda \text{Id}$ and consider the reduced manifold $M_A = (\{x \in \mathbb{R}^{2n+2} \mid \Omega(x, Ax) = 1\} / \{\exp tA\})_{cc}$ with the reduced symplectic structure ω^{red} and the reduced symplectic connection ∇^{red} .*

- *Each $(M_A, \omega^{red}, \nabla^{red})$ is a symmetric symplectic Ricci-type manifold;*

1. *it admits the group $\tilde{G}_A = \{B \in Sp(\mathbb{R}^{2n+2}, \Omega) \mid BA = AB\}_0$, as well as the quotient group $(\tilde{G}_A / \{\exp tA\})_0$, as groups of automorphisms acting transitively;*
2. *the action of \tilde{G}_A is strongly Hamiltonian : if D is an element in the Lie algebra $\tilde{\mathfrak{g}}_A$ of \tilde{G}_A and if D^{*M_A} denotes the corresponding fundamental vector field on M_A , i.e. $D_y^{*M_A} = \frac{d}{dt}|_0 \exp(-tD) \cdot y$ then $\iota(D^{*M_A})\omega^{red} = df_D$ with $(\pi^*(f_D))(x) = \frac{1}{2}(\Omega(x, Dx))$ and the associated moment map*

$$J : M_A \rightarrow \tilde{\mathfrak{g}}_A^* : \pi(x) \mapsto [D \rightarrow \frac{1}{2}(\Omega(x, Dx))]$$

is \tilde{G}_A equivariant.

3. *Any symplectic space form is diffeomorphic to a quotient of the universal cover of a model space $(M_A, \omega^{red}, \nabla^{red})$.*

A description of the model spaces appeared in [5]. We now refine this description as follows :

Proposition 1.3. Case 1: $A^2 = \lambda \text{Id}$ with $\lambda > 0$

If $\lambda = k^2, k > 0$ we view $\mathbb{R}^{2n+2} = L_+ \oplus L_-$ as the sum of the two Lagrangian subspaces corresponding to the $\pm k$ eigenspaces for A . In an adapted basis where $A = \begin{pmatrix} k \text{Id} & 0 \\ 0 & -k \text{Id} \end{pmatrix}$ and

$\Omega = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$, the level set is given by

$$\Sigma_A = \{(u, v) \in \mathbb{R}^{2(n+1)} \mid u \cdot v = -\frac{1}{2k}\}.$$

Seeing the cotangent bundle to the sphere as

$$T^*S^n = \{(\tilde{u}, \tilde{v}) \in \mathbb{R}^{2(n+1)} \mid \|\tilde{u}\| = 1, \tilde{u} \cdot \tilde{v} = 0\},$$

the canonical cotangent symplectic structure is given by the restriction to T^*S^n of the 2-form $\tilde{\Omega}$ on $\mathbb{R}^{2(n+1)}$ defined by $\sum_{i=1}^{n+1} d\tilde{v}^i \wedge d\tilde{u}^i$.

The map $\phi : \Sigma_A \rightarrow T^*S^n : (u, v) \mapsto (\tilde{u} = \frac{v}{\|v\|}, \tilde{v} = \|v\|u + \frac{1}{2k} \frac{v}{\|v\|})$, induces a diffeomorphism

$$\psi : M^{\text{red}} \rightarrow T^*S^n : \pi(u, v) = \pi(e^t u, e^{-t} v) \mapsto (\tilde{u} = \frac{v}{\|v\|}, \tilde{v} = \|v\|u + \frac{1}{2k} \frac{v}{\|v\|}).$$

The pullback by this diffeomorphism of the canonical symplectic structure on T^*S^n is the reduced symplectic structure ω^{red} on M^{red} ; indeed

$$\pi^*(\psi^*(\tilde{\Omega}_{|T^*S^n})) = \phi^*(\tilde{\Omega}_{|T^*S^n}) = \left(\sum_{i=1}^{n+1} du^i \wedge dv^i - \frac{d(u \cdot v) \wedge (v \cdot dv)}{\|v\|^2} \right)_{|_{\Sigma_A}} = \Omega_{|_{\Sigma_A}} = \pi^* \omega^{\text{red}},$$

and π^* is injective. Since T^*S^n is connected, $M_A = M^{\text{red}}$, and, as a symplectic manifold, we have

$$M_A \simeq T^*S^n. \quad (12)$$

Observe that it is simply connected when its dimension is at least 4.

A matrix B is in \tilde{G}_A if and only if it is of the form $B = \begin{pmatrix} C & 0 \\ 0 & {}^t C^{-1} \end{pmatrix}$ with $C \in Gl_+(n+1, \mathbb{R})$ so that $\tilde{G}_A \simeq Gl_+(n+1, \mathbb{R})$. The group $G_A = \tilde{G}_A / \{\exp tA\}$ identifies with the subgroup $Sl(n+1, \mathbb{R})$; it acts transitively on M_A , and the symmetric structure is defined by

$$M_A = Sl(n+1, \mathbb{R}) / Gl_+(n, \mathbb{R}) \quad (13)$$

with $Gl_+(n, \mathbb{R})$ sitting in $Sl(n+1, \mathbb{R})$ as $\left\{ \begin{pmatrix} \det B^{-1} & 0 \\ 0 & B \end{pmatrix} \mid B \in Gl_+(n, \mathbb{R}) \right\}$; it is the stabilizer of the

point $p_0 := \pi(-\frac{1}{2k}e_1, e_1)$ for $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$; the symmetry s_{p_0} is induced by $S_0 := \begin{pmatrix} 1 & 0 \\ 0 & -\text{Id} \end{pmatrix}$ and conjugation by S_0 is an automorphism σ of $Sl(n+1, \mathbb{R})$ with $Gl_+(n, \mathbb{R})$ the connected component of its fixed points.

The action of $Sl(n+1, \mathbb{R})$ is strongly Hamiltonian; identifying $\mathfrak{sl}(n+1, \mathbb{R})^*$ to $\mathfrak{sl}(n+1, \mathbb{R})$ via the Killing form, the associated moment map is

$$J' : M_A \mapsto \mathfrak{sl}(n+1, \mathbb{R}) : \pi(u, v) \mapsto \frac{1}{2(n+1)} \left(-u \otimes (v \cdot) + \frac{1}{n+1} (u \cdot v) \text{Id} \right);$$

it presents the homogeneous symplectic manifold M_A as a double cover of the adjoint orbit in $\mathfrak{sl}(n+1, \mathbb{R})$ of the element $\frac{1}{4k(n+1)^2} \begin{pmatrix} n & 0 \\ 0 & -\text{Id} \end{pmatrix}$; this orbit is $Sl(n+1, \mathbb{R}) / Gl(n, \mathbb{R})$.

Case 2: $A^2 = \lambda \text{Id}$ with $\lambda < 0$

If $\lambda = -k^2 < 0$, then $J = \frac{1}{k}A$ defines a complex structure, identifying \mathbb{R}^{2n+2} to \mathbb{C}^{n+1} . We denote by p the integer such that the signature of the quadratic form $\Omega(x, Ax)$ on $\mathbb{R}^{2(n+1)}$ is $(2(p+1), 2(n-p))$; the pseudo-Hermitian structure defined by $\langle x, y \rangle := \Omega(x, Jy) - i\Omega(x, y)$ on \mathbb{C}^{n+1} is of signature (over \mathbb{C}) equal to $(p+1, n-p)$. Clearly $0 \leq p \leq n$. The level set is given by $\Sigma_A = \{z \in \mathbb{C}^{n+1} \mid \langle z, z \rangle = \frac{1}{k}\}$ and the reduced space $M^{\text{red}} = \{[z] \mid z \in \mathbb{C}^{n+1}, \langle z, z \rangle = \frac{1}{k}, [z] = [e^{it}z]\}$ is connected and simply connected. A \mathbb{R} -linear endomorphism of $\mathbb{R}^{2(n+1)}$ which commutes with J identifies with a \mathbb{C} -linear endomorphism of \mathbb{C}^{n+1} . The group \tilde{G}_A identifies with the pseudo-unitary group $U(p+1, n-p) = Sp(\mathbb{R}^{2(n+1)}, \Omega) \cap Gl(n+1, \mathbb{C})$ and the quotient $\tilde{G}_A / \{\exp tA\}$ to a finite quotient (by $(n+1)$ th roots of the identity) of its subgroup $SU(p+1, n-p)$ and we denote in this case by $G_A = SU(p+1, n-p)$. The model space is

$$M_A = M^{\text{red}} = SU(p+1, n-p) / U(p, n-p), \quad (14)$$

endowed with its natural invariant pseudo Kähler structure, with $U(p, n-p)$ sitting in $SU(p+1, n-p)$ as $\left\{ \begin{pmatrix} \det B^{-1} & 0 \\ 0 & B \end{pmatrix} \mid B \in U(p, n-p) \right\}$ i.e. as the stabilizer of the point $p_0 := \pi(e_1)$ for $e_1 = (1, 0, \dots, 0) \in \mathbb{C}^{n+1}$. Thus

- $M_A = \mathbb{C}H^n = SU(1, n)/U(n)$ is the complex hyperbolic space for $p = 0$;
- $M_A = E$ is a holomorphic vector bundle of rank $n - p$ over $\mathbb{C}P^p$ for $1 \leq p \leq n - 1$;
- $M_A = \mathbb{C}P^n = SU(n + 1)/U(n)$ is the complex projective space for $p = n$.

The action of $SU(p + 1, n - p)$ is strongly Hamiltonian; using the trace to identify $\mathfrak{su}(p + 1, n - p)^*$ with $\mathfrak{su}(p + 1, n - p)$, the moment map is

$$J' : M_A \mapsto \mathfrak{su}(p + 1, n - p) : \pi(z) \mapsto \left(\frac{-i}{2} z \otimes \langle \cdot, z \rangle + \frac{i \langle z, z \rangle}{2(n + 1)} \text{Id} \right);$$

it presents the homogeneous symplectic manifold M_A as the adjoint orbit of $\frac{1}{2k(n+1)} \begin{pmatrix} -ni & 0 \\ 0 & i \text{Id} \end{pmatrix}$ in $\mathfrak{su}(p + 1, n - p)$.

Case 3: $A^2 = 0$

If $\lambda = 0$ there are two integers r and p attached to the space form, r being the rank of A (so that $1 \leq r \leq n + 1$) and $(p, r - p)$ being the signature of the quadratic 2-form $\Omega(x, Ax)$ (so that $1 \leq p \leq r$) which induces a pseudo Riemannian metric g on the quotient

$$V := \mathbb{R}^{2(n+1)} / \text{Ker } A.$$

Observe that $\text{Ker } A$ is the Ω -orthogonal of $\text{Im } A$, and $\text{Im } A \subset \text{Ker } A$ so that

$$W := \text{Ker } A / \text{Im } A$$

with the 2-form Ω' induced by Ω is a $2m := 2(n + 1 - r)$ -dimensional symplectic vector space. Choosing a r -dimensional Ω -isotropic subspace V' supplementary to $\text{Ker } A$, we have $V \simeq V'$ and we denote by g' the corresponding metric of signature $(p, r - p)$ on V' . If W' is the Ω -orthogonal to $AV' \oplus V'$, then $W' \simeq W$, and we write $\mathbb{R}^{2(n+1)} = (AV' \oplus V') \oplus^{\perp \Omega} W'$; in well chosen corresponding basis,

$$A = \begin{pmatrix} 0 & \text{Id}_r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} 0 & -G & 0 \\ G & 0 & 0 \\ 0 & 0 & \Omega' \end{pmatrix}, \text{ with } G := \text{Id}_p \oplus -\text{Id}_{r-p}, \text{ so the level set is}$$

$$\Sigma_A = \{(Ax, v, w) \mid x, v \in V', w \in W', g'(v, v) = \Omega(v, Av) = 1\}. \text{ We define}$$

$$Q^{p, r-p} := \{v \in V \simeq \mathbb{R}^r \mid g(v, v) = 1\}$$

and identify its tangent bundle to $TQ^{p, r-p} = \{v, y \in V \mid g(v, v) = 1, g(y, v) = 0\}$. The map

$$\phi : M^{\text{red}} \rightarrow T^*Q^{p, r-p} \times W : \pi(Ax, v, w) \mapsto (v, -g(x, \cdot), w)$$

is well defined since $\exp tA \cdot (Ax, v, w) = (Ax + tAv, v, w)$ and defines a symplectomorphism

$$M^{\text{red}} \simeq T^*Q^{p, r-p} \times W \tag{15}$$

with the product of the canonical cotangent bundle symplectic structure on $T^*Q^{p, r-p}$ and the natural symplectic structure on W . Remark that M^{red} is connected iff $p > 1$, and has two connected components if $p = 1$; it is simply connected when $p \neq 2$.

The group of symplectic transformations of $\mathbb{R}^{2(n+1)}$ commuting with A , written in the basis described above, is

$$\left\{ \begin{pmatrix} B & BGT & BU\Omega' \\ 0 & B & 0 \\ 0 & -C({}^{tr}U)G & C \end{pmatrix} = \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} \text{Id} & GT & U\Omega' \\ 0 & \text{Id} & 0 \\ 0 & -{}^{tr}UG & \text{Id} \end{pmatrix} \mid \begin{array}{l} B \in O(p, r - p), U \in \text{Mat}(r \times 2m, \mathbb{R}) \\ T = S + \frac{1}{2}GU\Omega'({}^{tr}U)G, \\ C \in Sp(2m, \mathbb{R}), S \in \text{Sym}(r, \mathbb{R}) \end{array} \right\}$$

so that \tilde{G}_A is the semidirect product of the semisimple subgroup S , which is the direct product $S := SO_0(p, r - p) \times Sp(2m, \mathbb{R})$, and the normal solvable subgroup $R = \text{Mat}(r \times 2m, \mathbb{R}) \cdot \text{Sym}(r, \mathbb{R})$ where $\text{Sym}(r, \mathbb{R})$ is the set of symmetric real $r \times r$ matrices, with the product

$$(U, S) \cdot (U', S') = (U + U', S + S' + \frac{1}{2}G(U'\Omega'({}^{tr}U) - U\Omega'({}^{tr}U'))G).$$

As symmetric space, we have

$$M_A = S \cdot R / (SO_0(p-1, r-p) \times Sp(2m, \mathbb{R})) \cdot (Mat((r-1) \times 2m, \mathbb{R}) \cdot (\mathbb{R} \times Sym((r-1), \mathbb{R}))) \quad (16)$$

which is the quotient of \tilde{G}_A by the stabilizer of $\pi(0, e_1, 0)$, with e_1 the first vector in the chosen pseudo-orthogonal basis of V , thus the set of elements of \tilde{G}_A mapping $(0, e_1, 0)$ on an element of the form $(te_1, e_1, 0)$ for some $t \in \mathbb{R}$, i.e. those elements (B, C, U, S) described above with $Be_1 = e_1$, ${}^{tr} Ue_1 = 0$ and $Se_1 = te_1$.

Denoting by (b, c, u, s) an element in the Lie algebra $\tilde{\mathfrak{g}}_A$, the moment map

$$J : M_A \simeq (T^*Q^{p, r-p} \times W)_{cc} \rightarrow \tilde{\mathfrak{g}}_A^*$$

reads $\langle J(\pi(Ax, v, w)), (b, c, u, s) \rangle = -g(x, bv) + \frac{1}{2}s(v, v) + g(v, U\Omega'w) + \frac{1}{2}\Omega(w, cw)$.

Hence $J(\pi(Ax, v, w)) = J(\pi(Ax', v', w'))$ iff $(Ax', v', w') = \pm(Ax, v, w)$, so the moment map is a double cover of the orbit of the element $\frac{1}{2}s_{11}^* \in \tilde{\mathfrak{g}}_A^*$ defined by $s^*(b, c, u, s) = s^{11}$ when $p > 1$. When $p = 1$, M_A is one of the connected component of $T^*Q \times W$ hence it corresponds to elements in $v \in Q$ so that the first component has a given sign, i.e. $v^1 > 0$ for one component and $v^1 < 0$ for the other component. In that case, if $\pi(Ax, v, w)$ is in M_A then $\pi(-Ax, -v, -w)$ is not in M_A so J is a symplectomorphism between M_A and the coadjoint orbit of $\frac{1}{2}s_{11}^*$ in $\tilde{\mathfrak{g}}_A^*$.

2 Homogeneous spaces in symplectic Radon duality

2.1 Totally geodesic symplectic submanifolds in M_A

We first show that our model spaces possess as many as possible symplectic totally geodesic submanifolds.

Theorem 2.1. *Let $M_A := (\Sigma_A / \{\exp tA\})_{cc}$ be a model of symplectic space form of dimension $2n$ as constructed above, with its canonical symmetric connection which is of Ricci type. We denote as before by ρ the corresponding Ricci endomorphism.*

1. *Let S be a totally geodesic symplectic submanifold of M_A , of dimension $2q$, passing through y and let $V = T_y S \subset T_y M_A$. Then V is a symplectic subspace of $T_y(M_A)$ stable by ρ_y .*
2. *Reciprocally, let V be a $2q$ -dimensional symplectic subspace of $T_y(M_A)$ stable by ρ_y . There exists a unique maximal totally geodesic submanifold S of M_A , of dimension $2q$, passing through y and tangent to V . It is given by*

$$S = (\Sigma_A \cap W / \{\exp tA\})_{cc} \quad \text{with} \quad W = \overline{V} \oplus \mathbb{R}x \oplus \mathbb{R}Ax$$

where x is a point in Σ_A so that $\pi(x) = y$ and where \overline{V} the $2q$ -subspace of \mathbb{R}^{2n+2} which is the horizontal lift of V in $T_x(\Sigma_A) = \text{Span}\{Ax\}^{\perp_\Omega} \subset \mathbb{R}^{2n+2}$, i.e. the subspace defined by $\Omega(\overline{V}, x) = 0$, $\Omega(\overline{V}, Ax) = 0$ and $\pi_{*x}\overline{V} = V$.

Such a totally geodesic submanifold is automatically a symplectic space form.

Proof. The proof follows from the fact that a submanifold S in a manifold endowed with a torsionfree connection (M, ∇) is totally geodesic if and only if, for any $y \in S$ and any $X \in T_y S$, $\nabla_X Y(y)$ belongs to $T_y S$ whenever Y is tangent to S .

In particular, if S is a totally geodesic submanifold $R_y^\nabla(X, Y)Z \in T_y S$ for any $y \in S$ and $X, Y, Z \in T_y S$. In view of the expression of a Ricci-type curvature given by (9), the tangent space $T_y S$ at the point $y \in S$ to a symplectic totally geodesic submanifold S in a Ricci-type

symplectic manifold (M, ω, ∇) is stable by ρ_y^∇ and this proves (1).

To prove (2), we observe that, for any $\tilde{x} \in W$ and any W -valued vector fields \tilde{X}, \tilde{Y} on W , then

$$\overset{\circ}{\nabla}_{\tilde{X}} \tilde{Y}(\tilde{x}) - \Omega(A\tilde{X}, \tilde{Y})\tilde{x} + \Omega(\tilde{X}, \tilde{Y})A\tilde{x} \quad \text{is in } W,$$

since the stability of V by ρ_y implies the stability of W by A . Using formula (11) which describes the reduced connection, $\nabla_X^{red} Y(y)$ belongs to $T_y S$ for all $y \in S$, $X \in T_y S$ and Y is tangent to S when $S = (\Sigma_A \cap W / \{\exp tA\})_{cc}$. \square

2.2 Spaces in Radon-type duality

The group $G_A := (\{B \in Sp(\mathbb{R}^{2n+2}, \Omega) \mid BA = AB\} / \{\exp tA\})_0$ acts by symplectic affine transformations on $(M_A, \omega^{red}, \nabla^{red})$; it maps a symplectic totally geodesic submanifold of dimension $2q$ on a symplectic totally geodesic submanifold of dimension $2q$.

Theorem 2.2. *Let $M_A := (\Sigma_A / \{\exp tA\})_{cc}$ be a model of symplectic space form of dimension $2n$ as described before.*

1. *There exists a finite number of orbits of G_A in the set of symplectic maximal totally geodesic submanifolds of M_A for any given dimension $2q$.*
2. *Each of these G_A -orbits is a symmetric space.*
3. *If $A^2 \neq 0$, those orbits are symplectic symmetric spaces.*

Proof. Point (1) follows from the fact that the action of G_A on the space of symplectic maximal totally geodesic submanifolds of M_A corresponds bijectively to the action of G_A on the set of $(2q+2)$ -dimensional symplectic subspaces W of \mathbb{R}^{2n+2} which are stable by A and intersect $\Sigma_A = \{x \in \mathbb{R}^{2n+2} \mid \Omega(x, Ax) = 1\}$.

To see point (2), one observes that given a subspace W , the conjugation by $\text{Id}_W \oplus -\text{Id}_{W^\perp}$ is an automorphism of \tilde{G}_A . It induces an automorphism of G_A . The fixed points of this automorphism are the elements which map W in W .

Point (3) results from the precise description given below. \square

We consider the Radon transform described in the introduction, choosing one orbit N of the automorphism group G_A in the set of symplectic maximal totally geodesic submanifolds of M_A . One associates to a continuous function f on M_A , with compact support, the function $\text{Rad } f$ on N defined by $\text{Rad } f(S) = \int_{x \in S} f(x) d\mu(x)$ with $d\mu$ an invariant measure on the totally geodesic submanifold S (which exists since S is a symplectic space form). The dual Radon transform associates to a continuous function F on N , with compact support, the function $\text{Rad}^* F$ on M_A , $\text{Rad}^* F(x) = \int_{S \ni x} F(S) d\nu(S)$ with $d\nu$ an invariant measure on N (which can be shown to exist from the explicit description below).

The spaces M_A and N which are in such Radon-type duality are the following :

Proposition 2.3. *The G_A -orbits of maximal totally geodesic symplectic submanifolds in the models spaces M_A , when $A^2 \neq 0$, are given as follows.*

- *When $A^2 = k^2 \text{Id}$, we view as before $\mathbb{R}^{2n+2} = L_+ \oplus L_-$ as a sum of two Lagrangian subspaces corresponding to the $\pm k$ eigenspaces for A . The group G_A is $Sl(n+1, \mathbb{R})$ and*

the model space form is the cotangent bundle to the sphere with its canonical symplectic structure

$$M = T^*S^n = Sl(n+1, \mathbb{R})/Gl_+(n, \mathbb{R}).$$

Any symplectic maximal totally geodesic submanifold of dimension $2q$ is diffeomorphic to

$$T^*S^q.$$

All such submanifolds are in the same orbit of $Sl(n+1, \mathbb{R})$; this orbit is given by

$$N_q = Sl(n+1, \mathbb{R})/S(Gl(q+1, \mathbb{R}) \times Gl(n-q, \mathbb{R})),$$

i.e. the space of pairs of supplementary spaces in \mathbb{R}^{n+1} , with one of the spaces of dimension $q+1$. It is a symmetric symplectic space.

- If $A^2 = -k^2 \text{Id}$, we view as before $A = kJ$ with J a complex structure and we identify \mathbb{R}^{2n+2} to \mathbb{C}^{n+1} which is endowed with the Hermitian structure $\langle u, v \rangle = \Omega(u, Jv) - i\Omega(u, v)$ with complex signature $(p+1, n-p)$.

1. When $p = n$, the group $G_A = SU(n+1)$ and the model space form is the complex projective space

$$M = P_n(\mathbb{C}) = SU(n+1)/U(n).$$

Every symplectic maximal totally geodesic submanifold of real dimension $2q$ is diffeomorphic to

$$P_q(\mathbb{C}).$$

There is only one $SU(n+1)$ -orbit of symplectic maximal totally geodesic submanifolds for a given dimension $2q$; it is given by

$$N_q = SU(n+1)/S(U(q+1) \times U(n-q)).$$

The Radon transform in the case where $q = n-1$ corresponds to the transform defined by antipodal submanifolds in $P_n(\mathbb{C})$ (see [9]).

2. When $1 < p < n$, the group G_A is $SU(p+1, n-p)$ and the model space form is

$$M = SU(p+1, n-p)/U(p, n-p).$$

Any symplectic maximal totally geodesic submanifold of real dimension $2q$ is of the form

$$SU(p'+1, q-p')/U(p', q-p')$$

for $p' < \min(p, q)$. There is only one $SU(n+1)$ -orbit of symplectic maximal totally geodesic symplectic submanifolds for a given dimension $2q$ and a given p' , $N_{q,p'}$ which is the symmetric symplectic space given by

$$N_{q,p'} = SU(p+1, n-p)/S(U(p'+1, q-p') \times U(p-p', n-p-(q-p'))).$$

3. When $p = 0$, the group G_A is $SU(1, n)$ and the model space form is the complex hyperbolic space

$$M = SU(1, n)/U(n) = H_n(\mathbb{C}).$$

Every symplectic totally geodesic submanifold of dimension $2q$ is diffeomorphic to

$$H_q(\mathbb{C}).$$

There is one $SU(1, n)$ -orbit of symplectic maximal totally geodesic symplectic submanifolds of a given dimension $2q$; it is given by

$$N = SU(1, n)/S(U(1, q) \times U(n - q)).$$

The Radon transform in the case where $q = n - 1$ corresponds to the Radon transform defined in the Riemannian framework with totally geodesic complex hypersurfaces in $H_n(\mathbb{C})$ (see again [9]).

- When $A^2 = 0$, we use as before the decomposition

$$\mathbb{R}^{2n+2} = (V \oplus V') \oplus^{\perp \Omega} W',$$

where $V = \text{Im}(A)$, V' is a Ω -isotropic subspace supplementary to $\text{Ker}(A)$ et W' is the symplectic subspace Ω -orthogonal to $V \oplus V'$. The group \tilde{G}_A is the following semi-direct product

$$S \cdot R = (SO_0(p, r - p) \times Sp(2m, \mathbb{R})) \cdot (Mat(r \times 2m, \mathbb{R}) \cdot Sym(r, \mathbb{R})),$$

and the model space is the symplectic product of a cotangent bundle, endowed with its canonical symplectic structure, and a symplectic vector space :

$$M^{red} = T^*Q^{p, r-p} \times W = (S \cdot R)/(S' \cdot R'),$$

where

$$\begin{aligned} S' &= (SO_0(p - 1, r - p) \times Sp(2m, \mathbb{R})), \\ R' &= (Mat((r - 1) \times 2m, \mathbb{R}) \cdot (\mathbb{R} \times Sym((r - 1), \mathbb{R}))). \end{aligned}$$

Any maximal symplectic totally geodesic submanifold of dimension $2q$ is diffeomorphic to

$$T^*Q^{p', r'-p'} \times W'.$$

There is only one $(S \cdot R)$ -orbit of symplectic maximal totally geodesic submanifolds for a given dimension $2q$, a given rank r' and a given signature p' , denoted by $N_{q, r', p'}$, which is the following symmetric space

$$N_{q, r', p'} = (S \cdot R)/(S'' \cdot R''),$$

where

$$\begin{aligned} S'' &= S(O(p', r' - p') \times O(p - p', (r - r') - (p - p')))_0 \\ &\quad \times (Sp(2m', \mathbb{R}) \times Sp(2(m - m'), \mathbb{R})), \\ R'' &= (Mat(r' \times 2m', \mathbb{R}) \times Mat((r - r') \times 2(m - m'))) \\ &\quad \cdot (Sym(r', \mathbb{R}) \times Sym(r - r', \mathbb{R})), \end{aligned}$$

with $2m' := 2q - 2r' + 2$.

3 Characterization in terms of totally geodesic submanifolds

We have seen in the previous section that our models Σ_A of space forms admit as many symplectic maximal totally geodesic submanifolds as possible. Reciprocally, we have :

Theorem 3.1. *Let (M, ω) be a symplectic manifold of dimension $2n \geq 8$. Assume (M, Ω) to be endowed with a symplectic connection ∇ and with a smooth parallel field A of endomorphisms of the tangent bundle such that, $A^2 = \lambda \text{Id}$ with λ constant and, for all $y \in M$, $A_y \in \mathfrak{sp}(T_y M, \omega_y)$. We assume here that the rank of A is at least 3.*

Assume that, for any point $y \in M$ and any symplectic A_y -stable subspace V of $T_y M$, there exists a totally geodesic submanifold S of M such that $y \in S$ and $T_y S = V$. Then (M, ω, ∇) is locally symmetric and its canonical connection is of Ricci-type.

Proof. Since the set of triples of tangent vectors to a point $y \in M$, $X, Y, Z \in T_y M$, such that $X, Y, Z, A_y X, A_y Y, A_y Z$ are linearly independent and span a symplectic A_y -stable subspace of $T_y M$, is a dense open subset of $(T_y M)^3$, the condition implies that the curvature is given by

$$R_y^\nabla(X, Y)Z = \alpha_y(Y, Z)X + \beta_y(X, Z)Y + \gamma_y(X, Y)Z + \alpha'_y(Y, Z)A_y X + \beta'_y(X, Z)A_y Y + \gamma'_y(X, Y)A_y Z$$

with $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ 2-forms on M .

Now $R_y^\nabla(X, Y) = -R_y^\nabla(Y, X)$ and $\bigoplus_{XYZ} R_y^\nabla(X, Y)Z = 0$ imply

$$\beta = -\alpha, \gamma(X, Y) = -\alpha(X, Y) + \alpha(Y, X), \beta' = -\alpha', \gamma'(X, Y) = -\alpha'(X, Y) + \alpha'(Y, X)$$

and the fact that $R_y^\nabla(X, Y)$ commutes with A_y since $\nabla A = 0$ implies

$$\alpha(Y, Z) = \alpha'(Y, AZ).$$

The connexion ∇ being symplectic, we also have $\omega(R^\nabla(X, Y)Z, T) = \omega(R^\nabla(X, Y)T, Z)$, i.e.

$$\begin{aligned} & \alpha'(Y, AZ)\omega(X, T) - \alpha'(X, AZ)\omega(Y, T) + (\alpha'(Y, AX) - \alpha'(X, AY))\omega(Z, T) \\ & + \alpha'(Y, Z)\omega(AX, T) - \alpha'(X, Z)\omega(AY, T) + (\alpha'(Y, X) - \alpha'(X, Y))\omega(AZ, T) \end{aligned}$$

is symmetric in Z, T , so that

$$\begin{aligned} & \alpha'(Y, AZ)\omega(X, T) - \alpha'(X, AZ)\omega(Y, T) + 2(\alpha'(Y, AX) - \alpha'(X, AY))\omega(Z, T) \\ & + \alpha'(Y, Z)\omega(AX, T) - \alpha'(X, Z)\omega(AY, T) - \alpha'(Y, AT)\omega(X, Z) \\ & + \alpha'(X, AT)\omega(Y, Z) - \alpha'(Y, T)\omega(AX, Z) + \alpha'(X, T)\omega(AY, Z) = 0. \end{aligned} \quad (17)$$

Choosing $T = AX$ and X ω -orthogonal to Y, AY, Z, AZ yields

$$\alpha'(Y, AZ)\omega(X, AX) = \alpha'(X, AX)\omega(Y, AZ).$$

Thus $\alpha'(Y, AZ) = f\omega(Y, AZ)$ and relation (17) becomes

$$\begin{aligned} & f\omega(Y, AZ)\omega(X, T) - f\omega(X, AZ)\omega(Y, T) \\ & + \alpha'(Y, Z)\omega(AX, T) - \alpha'(X, Z)\omega(AY, T) - f\omega(Y, AT)\omega(X, Z) \\ & + f\omega(X, AT)\omega(Y, Z) - \alpha'(Y, T)\omega(AX, Z) + \alpha'(X, T)\omega(AY, Z) = 0. \end{aligned}$$

Choosing $T = X$ and X ω -orthogonal to Y, AY, Z, AZ yields

$$\alpha'(Y, Z)\omega(AX, X) + f\omega(X, AX)\omega(Y, Z) + \alpha'(X, X)\omega(AY, Z) = 0$$

hence $\alpha'(Y, Z) = f\omega(Y, Z) + g\omega(Y, AZ)$ and relation (17) becomes

$$2g(\omega(Y, AZ)\omega(AX, T) - \omega(X, AZ)\omega(AY, T)) = 0,$$

hence $g = 0$, so that $\alpha'(Y, Z) = f\omega(Y, Z)$ and

$$\begin{aligned} R_y^\nabla(X, Y)Z &= f(y)(\omega_y(Y, AZ)X - \omega_y(X, AZ)Y \\ &\quad + \omega_y(Y, Z)A_y X - \omega_y(X, Z)A_y Y + 2\omega_y(X, Y)A_y Z) \end{aligned}$$

which states that the connection is of Ricci-type. Bianchi's second identity implies that f is a constant and the result follows. \square

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